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ERGODIC AFFINE TRANSFORMATIONS ARE LOOSELY BERNOULLI

BY

D. A. LIND

ABSTRACT

We prove that an ergodic affine transformation of a compact abelian group is loosely Bernoulli, that is, it can be induced from a Bernoulli shift.

By replacing Ornstein's $\bar{d}$ process metric in the definition of very weak Bernoulli with another metric related to Kakutani equivalence, Feldman [2] produced a class of processes he dubbed "loosely Bernoulli". An ergodic transformation is then loosely Bernoulli if all processes arising from it are loosely Bernoulli. Using Feldman's metric, Weiss developed in [9] a theory of Kakutani equivalence parallel to the isomorphism theory of Bernoulli shifts.

Zero entropy loosely Bernoulli transformations are mutually Kakutani equivalent, and include all ergodic translations on infinite compact groups. Bernoulli shifts are loosely Bernoulli. Hence so are ergodic automorphisms of compact groups, since they are isomorphic to Bernoulli shifts (see [3], or [5], [6], and [7]). Here we combine these algebraic examples of loose Bernoullicity.

Let G be a compact abelian group equipped with Haar measure and the Borel σ-algebra. Let S be a continuous, algebraic automorphism of G, and $a \in G$. The transformation $g \mapsto Sg + a$ is an *affine transformation* of G.

THEOREM 1. *Ergodic affine transformations of compact abelian groups are loosely Bernoulli.*

We prove more. Suppose that (X, μ) is a Lebesgue space, and that U is a measure-preserving transformation (hereafter called *map*) of X. Let $\alpha : X \to G$ be measurable. The map $U \times_\alpha S : X \times G \to X \times G$ defined by

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$((U \times_\alpha S)(x,g) = (Ux, Sg + \alpha(x))$ is the *skew product* of U with S by the skewing function α.

THEOREM 2. *Ergodic skew products of zero entropy loosely Bernoulli transformations with automorphisms of compact abelian groups are loosely Bernoulli.*

The proofs are essentially the same. The ingredients are the following.

Call an automorphism S *periodic* if $S^n = I$ for some $n > 0$.

THEOREM A. *Ergodic skew products of zero entropy loosely Bernoulli maps with periodic group automorphisms are loosely Bernoulli.*

The proof of Theorem A is the same as that given by Weiss [9, theor. 7.3] for skew products with the identity automorphism, since a periodic automorphism is an isometry for a suitably chosen translation-invariant metric.

THEOREM B. [9, cor. 4.8] *Loose Bernoullicity persists under inverse limits.*

THEOREM C. *A skew product of a map with an ergodic group automorphism is isomorphic to their direct product.*

We have shown that the isomorphism in Theorem C preserves the group fibers [3], and indeed, by solving a related functional equation, that it can be chosen to be a translation on each group fiber [4].

We will prove Theorem 2. After this we will indicate the proof of Theorem 1.

PROOF OF THEOREM 2. The thread that stitches the above results together is that a general group automorphism is an inverse limit of periodic automorphisms followed by a skew product with an ergodic group automorphism (see [1, ch. III, §4] or [3, theor. 9.2]).

Let Γ be the dual group of G, and T be the automorphism of Γ dual to S. Define

$$\Gamma_1 = \{\gamma \in \Gamma : (T - I)\gamma = 0\},$$

and inductively

$$\Gamma_n = \{\gamma \in \Gamma : (T^n - I)\gamma \in \Gamma_{n-1}\}$$

for $n \geqq 2$. Each Γ_n is T-invariant, so that $\{\Gamma_n\}$ is increasing, say to Γ_∞. Denote the annihilator of Γ_n in G by H_n, and of Γ_∞ by H. Then H_n and H are S-invariant closed subgroups of G with $H_n \searrow H$.

We claim that T is aperiodic on Γ/Γ_∞, i.e. the only element periodic under T is zero. For suppose that $(T^k - I)\gamma \in \Gamma_n$, where we may suppose that $k > 1$. Then $(T^{nk} - I)\gamma \in \Gamma_n \subset \Gamma_{nk-1}$, so that $\gamma \in \Gamma_{nk} \subset \Gamma_\infty$.

If K is an S-invariant closed subgroup of G, let S_K denote the restriction of S to K, $S_{G/K}$ the factor automorphism, and $\pi : G \to G/K$ the quotient homomorphism. If $U : X \to X$ and $\alpha : X \to G$, then $\pi\alpha : X \to G/K$, and $U \times_{\pi\alpha} S_{G/K}$ is a factor of $U \times_\alpha S$. Using a Borel cross section to π, it is easy to show that $U \times_\alpha S$ is a skew product of $U \times_{\pi\alpha} S_{G/K}$ with S_K (for details, see [3, §2]). Whenever we write $U \times_{\pi\alpha} S_{G/K}$, the symbol π will denote the quotient $G \to G/K$.

Assume that U is zero entropy loosely Bernoulli, and that $U \times_\alpha S$ is ergodic. Then the factor $U \times_{\pi\alpha} S_{G/H_1}$ is ergodic. Since T is periodic on Γ_1, S is periodic on the dual G/H_1. By Theorem A, $U \times_{\pi\alpha} S_{G/H_1}$ is loosely Bernoulli. It is easily checked to have zero entropy. Now T is periodic on Γ_2/Γ_1, so that S is periodic on the dual $(G/H_2)/(G/H_1) = H_1/H_2$. Since $U \times_{\pi\alpha} S_{G/H_2}$ is a skew product of $U \times_{\pi\alpha} S_{G/H_1}$ with S_{H_1/H_2}, another application of Theorem A shows that $U \times_{\pi\alpha} S_{G/H_2}$ is zero entropy loosely Bernoulli. Inductively, $U \times_{\pi\alpha} S_{G/H_n}$ is zero entropy loosely Bernoulli for $n \geqq 1$.

Since $H_n \searrow H$, $U \times_{\pi\alpha} S_{G/H}$ is the inverse limit of the $U \times_{\pi\alpha} S_{G/H_n}$, and is therefore loosely Bernoulli by Theorem B.

Now T is aperiodic on Γ/Γ_∞, so that S_H is ergodic on the dual H. Hence S_H is Bernoulli [3]. Since $U \times_\alpha S$ is a skew product of $U \times_{\pi\alpha} S_{G/H}$ with S_H, it is isomorphic by Theorem C to the direct product $(U \times_{\pi\alpha} S_{G/H}) \times S_H$. It is easy to prove that the direct product of a Bernoulli shift with a loosely Bernoulli map is loosely Bernoulli. Hence $U \times_\alpha S$ is loosely Bernoulli.

REMARKS. 1) To reduce this to affine transformations, replace X by a one-point space $\{x\}$, U by the identity map, and let $\alpha(x) = a$. Then $U \times_\alpha S$ is the affine map $g \mapsto Sg + a$. The proof of Theorem 1 is then the same as that of Theorem 2, except that some stage in the inverse limit may involve taking a skew product of a cyclic permutation of a finite set with a periodic group automorphism. Such maps are easily checked to be zero entropy loosely Bernoulli.

2) Recently, Dan Rudolph ([8] and oral communication) proved the much deeper result that ergodic skew products of positive entropy loosely Bernoulli transformations with the identity automorphism of a compact group are loosely Bernoulli. His proof easily extends to skew products with periodic automorphisms, and allows a proof of Theorem 2 for positive entropy base transformations. We preferred here to keep to the more elementary zero entropy case, since that is all we needed for affine transformations.

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Department of Mathematics
University of Washington
Seattle, Washington 98195 USA